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A note on learning chaotic sunspot equilibrium

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Summary. In this paper we prove convergence to chaotic sunspot equilibrium through two learning rules used in the bounded rationality literature. The first one shows the convergence of the actual dynamics generated by simple adaptive learning rules to a probability distribution that is close to the stationary measure of the sunspot equilibrium; since this stationary measure is absolutely continuous it results in a robust convergence to the stochastic equilibrium. The second one is based on the E-stability criterion for testing stability of rational expectations equilibrium, we show that the conditional probability distribution defined by the sunspot equilibrium is expectational stable under a reasonable updating rule of this parameter. We also report some numerical simulations of the processes proposed.

JEL Classification Numbers: C61, E32

1. Introduction

Convergence of the dynamics generated by learning processes is the subject of many papers in the intertemporal economy literature (Chatterji (1995), Evans and Honkapohja (1994,1995), Grandmont (1998), Guesnerie and Woodford (1991) and Woodford (1990)). They developed methods (adaptive, recursive, Bayesian) for learning different types of equilibria (steady states, cycles and sunspots equilibria) which are based on the feedback that the learning rule generates in the structural equations. In this note we show two types of convergence to a sunspot equilibrium having an absolutely continuous invariant measure. The first one shows an adaptive learning process which defines with the structural equations an actual complex dynamics that mimics the stochastic characteristics of the stationary sunspot equilibrium, namely the recurrent behaviour of the actual dynamics defines a probability distribution which is close to the stationary measure of the sunspot equilibrium; in this sense we can say that the learning rule "converges" to the sunspot equilibrium. The second type of convergence results from the expectational stability of the conditional probabilities of the sunspot equilibrium. The expectational stability of a rational expectations equilibrium (Evans and Honkapohja (1995), Guesnerie (1993)) is used as a selection criterion of that type of equilibria. Specifically, this concept characterizes the stability of some parameters which define the rational expectation equilibrium under instantaneous corrections by some specific rule and the structural equations.

The types of sunspot equilibria we will consider in this work were proved to exist in Araujo and Maldonado (2000) and they typically exist when the backward perfect prevision map exhibits ergodic chaos (see Majumdar and Mitra (1994)) it means that there exists an absolutely continuous invariant and ergodic measure for it. Sometimes this measure is called "empirical", "observable" or "physical" (de Melo and van Strien (1993)). Here we will describe the main results on the existence of these types of sunspot equilibria (that we will call chaotic sunspot equilibria) and show how the invariant measures can be estimated.

Woodford (1990) proposed another learning rule based on the algorithm given by Ljung and Soderstrom (1983); with this learning rule he obtained convergence to the (finite) support of the considered SE from any initial state close to this support, so agents can learn the (support of the) sunspot. In this paper we will prove that for almost all initial state in a large interval: *i)* the agents can learn the stationary probability measure of the sunspot when they use simple adaptive learning rules and *ii)* they can learn the conditional probability distribution of that equilibrium if updating is in notional or virtual time. In this sense this paper will provide a theoretical fundation for considering a chaotic deterministic equilibrium as being a stochastic one (Hommes, van Strien and de Vilder (1994), de Vilder (1996)).

This paper is divided in seven sections. In section 2 we present the general framework and show the conditions for existence of chaotic sunspot equilibrium. In section 3 we make a review of one-dimensional dynamical systems for unimodal maps which will be used in the next section. In section 4 we show OLG models with the type of sunspot equilibrium constructed in section 2. In section 5 we show a learning process that generates an actual complex dynamics which mimics the SSE and in section 6 we prove the expectational stability of the conditional probabilities of the chaotic sunspot equilibrium. Conclusions are given in section 7 and the proofs are contained in the appendix.

2. Chaotic sunspot equilibrium

Let $X \subset \Re^n$ be the state variable set and $\mathcal{B}(X)$ denote the Borel sets of X. The equilibrium condition of our model is represented by the zeros of the function:

$$\tilde{Z} : X \times \mathcal{P}(X) \to \Re^n ,$$

where $\mathcal{P}(X) = \{ \mu : \mathcal{B}(X) \to [0,1] / \mu$ is a probability measure$\}$. We will call this map a *stochastic excess demand function* because in some models $\tilde{Z}(x_0, \mu)$ will be the excess demand when the present state variable is x_0 and the probability measure for the future state variable is μ. Most of the overlapping generations (OLG) models have this structure and we are allowing for no perfect prevision in general.

The *deterministic excess demand function* is: $Z : X \times X \to \Re^n$ defined by $Z(x_0, x_1) = \tilde{Z}(x_0, \delta_{x_1})$.

In models that admit a representative agent there exists some type of linearity in the rationalizing measures in the sense that they form a convex set. For these cases we have the next

(CVR) Property: The stochastic excess demand function $\tilde{Z}$ has the *convex valuedness of rationalizing measures* (CVR) property if $\forall\, x \in X$, $\forall\, \mu_1, \mu_2 \in \mathcal{P}(X)$ such that $\tilde{Z}(x, \mu_1) = \tilde{Z}(x, \mu_2) = 0$ and $\forall\, \alpha \in [0, 1]$ we have $\tilde{Z}(x, \alpha\mu_1 + (1 - \alpha)\mu_2) = 0$.

Since we will look for Markovian equilibria, let us remember that a transition function defined on X is a function $Q : X \times \mathcal{B}(X) \to [0, 1]$ such that: i) $\forall\, x \in X\ Q(x, .) \in \mathcal{P}(X)$ and ii) $\forall\, A \in \mathcal{B}(X)\ \ Q(., A)$ is a measurable function.

Definition 2.1.- A *sunspot equilibrium* (SE) is a pair (X_0, Q) where $X_0 \subset X$ and Q is a transition function on X_0 such that:

i) $\exists\, x_0 \in X_0$ such that $Q(x_0, .)$ is not a Dirac measure (it is truly stochastic).

ii) $\forall\, x \in X_0\ \tilde{Z}(x, Q(x, .)) = 0$.

We are following the Chiappori and Guesnerie (1991) structure. They compare this definition with the standard version of the sunspot equilibrium concept. Woodford (1986) presents another form for the sunspot equilibrium since his excess demand depends on "theories" from the total history of the extrinsics.

Definition 2.2.- A sunspot equilibrium (X_0, Q) is called *stationary* (SSE) if there exists $\mu \in \mathcal{P}(X)$ with support equal to X_0 such that

$$\mu(A) = \int_{X_0} Q(x, A)\, \mu(dx) \quad \forall\, A \in \mathcal{B}(X_0).$$

So, if a SE is stationary then the stochastic process generated by the measure μ and the transition function Q is a stationary Markov process.

In order to make the conection beetween the existence of SSE with a positive Lebesgue measure support and the chaoticity of the deterministic economy, we will give the following definition.

Definition 2.3.- A *backward perfect foresight* (bpf) map is a function $\phi : X \to X$ such that: $\quad Z(\phi(x), x) = 0\ ,\ \forall\, x \in X$.

This definition was also used by Grandmont (1986). It is easy to see that if $(x_t)_{t \geq 0}$ is a sequence such that $x_t = \phi(x_{t+1})$ $\forall t \geq 0$ then it is a perfect foresight equilibrium. The following theorem shows that if the bpf map has ergodic chaos then there exists a SSE for the economy.

Theorem 2.4. *Suppose that $\tilde{Z}$ has the CVR property and the bpf map $\phi : X \to X$ (where $X = [0, a]$) is a unimodal map with $\phi(0) = 0$. Let X_1 (X_2) be the interval where ϕ is strictly increasing (decreasing) and suppose that the maps $f : X \to X_1$ and $g : X \to X_2$ are the*

C^1 local inverses of ϕ. If $\exists \mu \in \mathcal{P}[0,1]$, $\mu << \lambda$ such that μ is ϕ-invariant, then we have that the following process is a SSE:

$$Q(x,A) = \frac{d\mu \circ f}{d\mu}(x)\delta_{f(x)}(A) + \frac{d\mu \circ g}{d\mu}(x)\delta_{g(x)}(A) \ \ , x \in supp(\mu).$$

The proof of theorem 2.4 in a more general setting is given in Araujo and Maldonado (2000) but we will prove this version in the appendix because we will use it in the following sections.

Remarks: If the measure μ is ergodic, the Birkhoff theorem implies:

$$\mu_N = \frac{1}{N}\sum_{n=0}^{N-1} \delta_{\phi^n(x)} \underset{N\to\infty}{\longrightarrow} \mu, \ \ \mu - a.e.\, x,$$

(the convergence is in the weak topology). Therefore with the backward perfect foresight trajectory from μ-almost initial state we can obtain the histograms corresponding to the measures μ_N (the empirical measures) and they will approximate the density (Radon-Nikodym derivative) of μ with respect to the Lebesgue measure (when $\mu << \lambda$). These histograms are constructed by taking partitions $(I_i)_{0\le i\le n}$, so an approximation of $\mu(I_i)$ is:

$$\mu_N(I_i) = \frac{\#\{\, j\,;\, x_j \in I_i \,,\, 0 \le j \le N\,\}}{N+1}.$$

3. Bowen-Ruelle-Sinai measures for unimodal maps

This section is devoted to describe in which cases a unimodal map has an absolutely continuous invariant measure. Let $X = [0,a]$ be a non-trivial interval.

A map $\phi : X \to X$ is called *unimodal* if ϕ has a unique interior local (maximum) extremum.

If c is the local extremum of a unimodal map ϕ, we will call it *non-flat* if there exists a C^2 local diffeomorphism h such that $h(c) = 0$ and $\phi(x) = \phi(c) \pm |h(x)|^\alpha$, for some $\alpha \ge 2$.

For example, if ϕ is C^∞ with some derivative non-zero at c then c is a non-flat critical point.

The *Schwarzian derivative* of $\phi \in C^3$ is defined by:

$$S\phi(x) = \frac{\phi'''(x)}{\phi'(x)} - \frac{3}{2}(\frac{\phi''(x)}{\phi'(x)})^2, \quad \text{if } \phi'(x) \ne 0.$$

Let us consider the following set of functions:

$$\mathcal{F} = \{\, \phi \in C^3(X);\ \ \phi \text{ is a unimodal map with non-flat extremum}\, c, S\phi < 0,\ \phi(0) = 0,\ \phi'(0) > 1\}$$

Given a function $\phi : X \to X$, we define the *ω-limit set* of the orbit from $x \in X$ as:

$$\omega(x) = \{\, y \in X; \text{there exists a subsequence } n_i \to \infty \text{ with}\, \phi^{n_i}(x) \to y\,\}.$$

In other words it is the set of accumulation points of the sequence $\{\phi^n(x)\}_{n\ge 0}$. The next theorem characterizes the ω-limit set of Lebesgue almost all points of the interval. It is proved in Blokh-Lyubich (1990).

Theorem 3.1. *If $\phi \in \mathcal{F}$ then there exists a unique set $A \subset X$ such that $\omega(x) = A$ $\lambda-$ a.e. x. Moreover A is either a finite union of intervals or has Lebesgue measure zero. Furthermore, if ϕ has an attracting periodic orbit then A is this periodic orbit.*

As noted in section 2, if the measure is ergodic we can use the backward trajectory to estimate the stationary distribution. We will in fact work with the concept of a "visible" measure (also called a Bowen-Ruelle-Sinai measure) where the backward trajectory can be used to estimate the stationary distribution from initial values in some set of positive Lebesgue measure.

Definition 3.2.- Let $\phi : X \to X$ and μ a ϕ-invariant probability measure. We say that μ is a *Bowen-Ruelle-Sinai (B-R-S) measure* if there exists $B \subset X$ with $\lambda(B) > 0$ such that for all $x \in B$:

$$\frac{1}{N} \sum_{n=0}^{N-1} \delta_{\phi^n(x)} \underset{N\to\infty}{\longrightarrow} \mu \text{ in the weak topology}$$

i.e. for all $F \in C^0(X)$: $\frac{1}{N} \sum_{n=0}^{N-1} F(\phi^n(x)) \underset{N\to\infty}{\longrightarrow} \int_X F(x)\mu(dx)$.

The difference between ergodic and B-R-S measures is that for the former the Birkhoff theorem holds just for the elements in the support of the measure and it can have Lebesgue measure zero (for example, if the support of the measure is a cycle) on the other hand B-R-S measures satisfies the ergodic (Birkhoff) theorem for a set with positive Lebesgue measure (a visible set).

The next theorem gives conditions for the existence of B-R-S measures of functions in $\mathcal{F}$. It is proved in de Melo-van Strien (1993).

Theorem 3.3. *if $\phi \in \mathcal{F}$, then:*

1) There is at most one B-R-S measure.

2) If $\nu << \lambda$ and ν is ϕ-invariant then:

i) ν is a B-R-S measure for ϕ and $\lambda(B) = \lambda(X)$ (B as in definition 3.5).

ii) The unique set A given in theorem 3.4 is a finite union of transitive intervals (J is a transitive interval if there exists $N > 0$ such that $\phi^N(J) \cap J \neq \{\}$).

iii) The support of ν is equal to A and ν is equivalent to $\lambda|_A$.

The second part of the theorem above says that it is sufficient to find an invariant and absolutely continuous (with respect to the Lebesgue measure) measure for obtaining a B-R-S measure which is supported in a union of transitive intervals. The following theorems state conditions for existence of absolutely continuous invariant measures.

Keller (1990) proved that there exists an absolutely continuous invariant probability measure if and only if ϕ has a positive Liapounov exponent in almost every point and in this way related the existence of such measures with the "chaoticity" of ϕ.

Theorem 3.4. *If $\phi \in \mathcal{F}$ then there exists $\lambda_\phi \in \Re$ such that for almost every x:*

$$\lambda_\phi = \limsup \frac{1}{n} \log |D\phi^n(x)|$$

Furthermore:

1) $\lambda_\phi > 0 \;\Leftrightarrow\; \phi$ *has an absolutely continuous invariant probability measure and* $\lambda_\phi = \lim \frac{1}{n}\log|D\phi^n(x)|$.

2) $\lambda_\phi < 0 \;\Leftrightarrow\; \phi$ *has an hyperbolic periodic attractor.*

In the following section we will show that a class of OLG models has unimodal bpf maps associated with its stochastic excess demand and we analyze in which cases it has B-R-S measures for obtaining the SSE.

4. An OLG economy with chaotic SE

In this section we will show a specific economy that exhibit chaotic SE for some set of parameters. We will consider the overlapping generations model with money transfer, subsidies and public expenditures treated in Grandmont (1986). The agents live two periods and have separable utility function $V_1(c_1) + V_2(c_2)$ where c_t is the consumption of the unique good at age $t = 1, 2$ and $V_i(c) = c^{1-\alpha_i}/(1-\alpha_i)$, $\alpha_i > 0$, $i = 1, 2$. Suppose that one unit of the good is produced with one unit of the unique productive factor (labor). The agent's endowments at each age $t = 1, 2$ are $l_1^* > 0$ and $l_2^* > 0$ and let $\bar{\theta} = V_1'(l_1^*)/V_2'(l_2^*)$. Define:

$$s_t = \frac{M_{t-1}z_t + S_t}{M_{t-1}z_t}\ ,\ d_t = \frac{M_{t-1}z_t + S_t + G_t}{M_{t-1}z_t}.$$

where $M_{t-1} > 0$ represents the money stock at the end of the period $t-1$, z_t is the money transfer factor ($z_t - 1$ is the nominal interest rate), S_t is the subsidy and G_t is the amount of money issued when the government purchases (or sells) some quantity of the good. The dynamic of the money supply is given by the equation:

$$M_t = M_{t-1}z_t + S_t + G_t \quad \text{or}$$

$$M_t = M_{t-1}z_t d_t, \quad M_0 \text{ given.}$$

s_t and d_t are exogenous variables and we suppose $s_t = s$, $d_t = d$ for all t.

Now let us analyze the agent's decision problem. Let p_t be the price of the good in period 1 and p_{t+1} the (random) price of the good in period 2. Then the agent must chose consumption plans c_t (deterministic), c_{t+1} (random) and money demand m so as to maximize:

$$V_1(c_t) + E[V_2(c_{t+1})]$$

with the budget constraints:

$$p_t c_t + m = p_t l_1^*$$

$$p_{t+1}c_{t+1} = p_{t+1}l_2^* + mz_t + S_{t+1}.$$

The first order condition for this problem (when the money demand is positive) is:

$$\frac{1}{p_t}V_1'(l_1^* - \frac{m}{p_t}) = E[\frac{z_{t+1}}{p_{t+1}}V_2'(l_2^* + \frac{mz_{t+1} + S_{t+1}}{p_{t+1}})] \tag{1}$$

the monetary equilibrium condition is $m = M_t$ and if we denote:

$$x_t = \frac{M_t}{p_t} \ , \ v_1(x) = xV_1'(l_1^* - x) \ , \ v_2(x) = xV_2'(l_2^* + x),$$

then we will have the equation (1) equivalent to:

$$v_1(x_t) = E[s^{-1}v_2(sd^{-1}x_{t+1})]$$

(we have used $M_t z_{t+1} = d^{-1}M_{t+1}$ and therefore $M_t z_{t+1} + S_{t+1} = sd^{-1}M_{t+1}$).

Remember that the expected value is taken with respect to the probability measure of x_{t+1} (or p_{t+1}) which is the agent's expectations of the future prices. In this case the excess demand function is:

$$\begin{aligned}\tilde{Z}(x,\mu) &= v_1(x) - E_\mu[s^{-1}v_2(sd^{-1}x')] \\ &= x(l_1^* - x)^{-\alpha_1} - E_\mu[d^{-1}x'(l_2^* + sd^{-1}x')^{-\alpha_2}],\end{aligned}$$

and it has the (CVR) property given in definition 2.2. The backward perfect foresight map is $\phi(x) = v_1^{-1}(s^{-1}v_2(sd^{-1}x))$, since v_1 is strictly increasing. In the following lemma we give the parameter set that guarantees $\phi \in \mathcal{F}$ (see section 3).

Lemma 4.1. *If $d\bar{\theta} < 1$, $\alpha_1 \in (0,1]$ and $\alpha_2 \in [2,+\infty)$ then $\phi \in \mathcal{F}$.*

Finally, in order to determine if there exists an invariant B-R-S measure (see definition 3.2) we have to test the condition given in theorem 3.4. The following theorem shows how it can be done.

Theorem 4.2. *If the hypothesis of lemma 4.1 holds and $\lambda_\phi = \limsup \frac{1}{n}|D\phi^n(x)| > 0$, $\lambda - a.e$ then there exists SSE whose invariant measure is an absolutely continuous B-R-S measure.*

The proof of this theorem follows from theorem 2.4 and theorem 3.4. The table bellow gives estimates of λ_ϕ for some values of α_1 and α_2 and the corresponding histograms of the function ϕ. The interval in the x-axis where the dynamics takes place is $[0, x^*]$. In this interval we take a partition and estimate the frequency of any orbit in each interval. The vertical axis shows these frequencies. So the histograms show an approximation for the density of the invariant measure μ. Also we can note that for $\lambda_\phi > 0$ the support of the invariant measure is all the interval whereas for $\lambda_\phi < 0$ the support is an attractive cycle. In the second case there exists a finite SSE close to the cycle as proved by Azariadis and Guesnerie (1986).

5. The convergence of a learning process to the chaotic SSE

In Araujo and Maldonado (2000) it was proved that the chaotic sunspot equilibrium can be learned by agents if: *i)* They know the deterministic characteristics of the economy (functions f and g) and the state variable is following the perfect foresight path given by the dynamical system $x_t = \phi(x_{t+1})$; in such a case they update expectations putting

$\mu_{t+1} = R^t(z)\delta_{f(z)} + (1-R^t(z))\delta_{g(z)}$ where $R^t(z)$ is the relative frequency in any subinterval of the points in the path $(x_0, ..., x_{t-1}, x_t = z)$ wich remains in I_1 (interval where ϕ is increasing); it results that $R^t(z) \to d\mu \circ f(z)/d\mu$ almost surely when $t \to \infty$. *ii)* The economy is following the stationary equilibrium given by the chaotic SSE, i.e. if the state variable is a random process $(\tilde{x}_t)_{t\geq 0}$ given by (Q, μ) ($Prob[\tilde{x}_{t+1} \in A | \tilde{x}_t = x] = Q(x, A)$ and $Prob[\tilde{x}_t \in A] = \mu(A)$); in this case the convergence of $R^t(z)$ results using the stochastic ergodic theorem.

In this section we will provide an adaptive learning processes that allows convergence to the chaotic SSE in the sense that the actual dynamics mimics the chaotic SSE. Woodford (1990) proposed an adaptive learning process based on the stochastic approximation algorithm of Robins and Monro (1951) for obtaining convergence to the support of the finite SE he considered. It is worth noting that such a method uses the deterministic structure of the model and the "projection facility" described in Marcet and Sargent (1989) for obtaining this convergence, so Woodford's method converges to a finite SE for any initial condition in a (probably small) neighborhood of the steady state, while our learning processes converge to a SE whose support has positive Lebesgue measure for any initial condition in a total Lebesgue measure set.

Also we provide a proof of the expectational stability (ES) of the conditional probabilities of the chaotic SSE. Expectational stability of some parameter which defines a rational expectation equilibrium means that this equilibrium is stable for the dynamics generated by the instantaneous corrections (in notional or virtual time) of this parameter. Such a concept was used for example by Evans and Honkapohja (1995) and Guesnerie (1993) in different contexts but the idea is the same: If we suppose some initial value of the parameter (which can be constructed in any specific way) the effect in the structural equations will give us a correction of it. If the dynamics defined in this way converges to some rational expectation equilibrium then we will say that such an equilibrium is expectationaly stable.

First, let us define the dynamics generated by learning processes (see for example Grandmont (1998)). Let X be the state variable set. A learning process is a sequence $(\psi_t)_{t\geq 1}$ such that $\psi_t : X^t \to \mathcal{P}(X)$ and it can be interpreted in the following way: If the economy followed the path $(x_0, ..., x_{t-1})$ then in period t agents will formulate expectations about the state variable in period $t+1$ with $\psi_t(x_0, ..., x_{t-1})$.

The structural equation is $\tilde{Z}(x_t, \mu_{t+1}) = 0$, so the actual dynamics under the learning process $(\psi_t)_{t\geq 1}$ is given by:

$$\tilde{Z}(x_t, \psi_t(x_0, ..., x_{t-1})) = 0$$

Let $\phi : [0, a] \to [0, a]$ be the bpf map associated to $\tilde{Z}$.

Learning process converging to chaotic SSE:

Suppose the following learning process: $\psi_t(x_0, ..., x_{t-1}) = \delta_{x_{t-1}}$. In this learning process agents update expectations in a very simple way, they put the probability of the next period state concentrated in the last observation. This can be seen as a limit of the process $x^e_{t+1} = \alpha x_{t-1} + (1-\alpha)x^e_{t-1}$ when $\alpha \to 1$. In this case the actual dynamics follows the backward trajectory from x_0 (the initial state), i.e. the actual dynamics holds $x_t = \phi(x_{t-1})$.

If $\phi \in \mathcal{F}$ and its Liapounov exponent is positive then most of the orbits associated are not stable (in the sense that they do not converge to some cycle). For an interval $I \subset [0,a]$ we can define the *average time* of a trajectory starting from x_0 in I as:

$$\tau_{x_0,N}(I) = \frac{\sharp\{i/\phi^i(x_0) \in I,\ i = 0, ..., N-1\}}{N}$$

So the following theorem shows that this economy with the learnig process above behaves like a SSE described in theorem 2.4.

Theorem 5.1. *Suppose that the bpf map $\phi \in \mathcal{F}$ and it has a positive Liapounov exponent for almost every initial state. Then the average time of any actual trajectory in any $I \subset [0,a]$ generated by the adaptive learning process $x^e_{t+1} = x_{t-1}$ converges to the $Prob[\tilde{x}_t \in I]$ where $(\tilde{x}_t)_{t\geq 0}$ is the stationary process generated by Q and μ of theorem 2.4.*

All the numerical simulations showing in the following figures were made for the model given in section 4 with the following parameters: $\alpha_1 = 0.21$, $\alpha_2 = 6.5$, $l^*_1 = 3.51$, $l^*_2 = 0.55$ and $s = d = 1$. Figure 1 shows the histogram defined by the invariant measure μ, it is obtained by the actual dynamics generated by the structural equation $\tilde{Z}(x_t, \mu_{t+1}) = 0$ and the learning process $x^e_{t+1} = x_{t-1}$. Figures 2 and 3 show the actual dynamics using the learning rule $x^e_{t+1} = \alpha x_{t-1} + (1-\alpha)x^e_{t-1}$ for $\alpha = 0.997$ and 0.99 respectively. It is worth noting that the frequencies are very sensitive to variations from $\alpha = 1$. Finally figure 4 results from using the following learning rule $x^e_{t+1} = \alpha_t x_{t-1} + (1-\alpha_t)x^e_{t-1}$ for $\alpha = 1-(1/t)$. It is also computed the Liapounov exponent and the L^1 distance between the density with $\alpha = 1$ and the densities for the other cases.

6. Expectational stability of chaotic SSE

Suppose that agents know the deterministic characteristics of the economy, it means that if there is no uncertainty with respect to the next period state x_{t+1} then the agents will choose $x_t = \phi(x_{t+1})$ as an optimal decision which equilibrates the market for the good. But if ϕ is a unimodal map then there exists another x'_{t+1} such that $x_t = \phi(x'_{t+1})$. In an uncertainty world the individuals would like to know the probabilities of x_{t+1} and x'_{t+1} because these states will make the individuals to choose x_t. Specifically, for each $x \in C$ (C with positive Lebesgue measure) the agents want to know $\alpha(x) \in (0,1)$ such that $(x, \alpha(x)\delta_{f(x)} + (1-\alpha(x))\delta_{g(x)})$ is a temporary equilibrium. Of course, any value of $\alpha(x)$ will give us the desired result, but we will show that $\alpha(x) = d\mu \circ f(x)/d\mu$ is expectational stable if the updating of these conditional probabilities is made in the following reasonable way. Let us remember that X_1 and X_2 are the subintervals of $[0,a]$ where ϕ is monotone. Let $(J_n)_n$ be a regular partition of $[0,a]$.

In period one, the only information that agents have is x_0, then the expectation for period 2 is made by $\psi_2 = \delta_{x_0}$. This expectation, when substituted in the structural equation $v_1(x_1) = E_{\mu_2}[v_2(x_2)]$ results in $x_1 = \psi(x_0)$. By induction, with the following information $\mathbf{x} = (x_0, x_1, ..., x_{t-1})$ individuals take as expectations for period $t+1$,

$$\psi_t(\mathbf{x}) = \pi(\mathbf{x})\delta_{x_{t-1}} + (1-\pi(\mathbf{x}))\delta_{x_{t-1}+\tilde{\epsilon}_t} \tag{2}$$

where $(\tilde{\epsilon}_t)_t$ is an independent and identically distributed random sequence with zero mean and support ϵ and the probabilities are given by:

$$\pi(\mathbf{x}) = \frac{\sharp\{i/x_i \in J_n,\ x_{i-1} \in X_1,\ i = 0, ..., t-2\}}{\sharp\{i/x_i \in J_n,\ i = 0, ..., t-1\}},\ x_0 \in J_n \tag{3}$$

It must be interpreted as the relative frequency of the points in the path $\mathbf{x} = (x_0, x_1, ..., x_{t-1})$ which come from the interval X_1. It is important to note that in this case (as in the usual analysis of expectational stability) t is a notional or virtual time, so what agents want to know is the conditional probabilities of $\delta_{f(x_0)}$ and $\delta_{g(x_0)}$.

Theorem 6.1. *Suppose that the bpf map given in section 4* $\phi = v_1^{-1}(s^{-1}v_2(sd^{-1}.)) \in \mathcal{F}$ *and it has a positive Liapounov exponent for almost every initial state. Let* $(J_n)_n$ *be a regular partition of* $[0, a]$ *and* $(\tilde{\epsilon}_t)_t$ *an independent and identically distributed random sequence with zero mean and support* ϵ*. Then the SSE* (μ, Q) *is expectational stable if the corrections of the conditional probabilities are defined by (2) and (3) in the sense that* $\pi(\mathbf{x}) \to \frac{d\mu\circ f}{d\mu}(x_0)$ *when* $t \to \infty$, $\epsilon \to 0$ *and the norm of the partition* $(J_n)_n$ *converges to zero for all* x_0 *in a total Lebesgue measure subset of* $[0, a]$.

The theorem 6.1 shows we can obtain convergence to the chaotic SSE in the sense of expectational stability (Evans and Honkaphoja (1995)). Figure 5 shows the function $d\mu \circ f/d\mu$ calculated from the measure μ and the function f and figures 6, 7 and 8 shows the "limits" of the π sequences for $\epsilon = 0, 0.001$ and 0.01 respectively. It is also reported the L^1 distance between these functions.

7. Conclusions

In this work we show two different ways of obtaining convergence to what we call a chaotic sunspot equilibrium. First of all we do an exposition of this type of sunspot equilibrium and we give conditions for its existence. After this we consider a class of overlapping generations models that can exhibit chaotic sunspot equilibrium.

In the last section we provide two stability results of the chaotic SSE. The first one shows that the actual dynamics generated by a simple adaptive learning rule lead almost all actual trajectory to a chaotic path which describes the stationary equilibrium given by the chaotic SSE. It was made when the gain of past observation is one but we provide some numerical examples showing that it holds when the gain is very close to one. In this sense such a learning rule can serve as a theoretical justification of how complex learning equilibria can mimic stochastic equilibnria (Christiano and Harrison (1996), de Vilder (1996)).

The second one proves the expectational stability of the chaotic SSE, it means that instantaneous corrections of the conditional expectations converges to the conditional probability of the chaotic SSE. We can say that both results are robust in the sense that the convergence is for almost all initial point in the support of the SSE which has a positive Lebesgue measure.

APPENDIX

Proof of theorem 2.4. Let us see that $\mu \circ f$ and $\mu \circ g$ are absolutely continuous with respect to μ and $d\mu \circ f/d\mu + d\mu \circ g/d\mu = 1$ for all x, $\mu - a.e.$.

If $B \in \mathcal{B}([0,a])$ is such that $\mu(B) = 0$ then $\mu(\phi^{-1}(B) = 0$. But $\phi^{-1}(B) = f(B) \cup g(B)$, then $\mu \circ f(B) = \mu \circ g(B) = 0$. Also if $A \in \mathcal{B}([0,a])$ the we have that $\mu(A) = \mu \circ f(A) + \mu \circ g(A)$ and:

$$\mu \circ f(A) = \int_A \frac{d\mu \circ f}{d\mu}(x)\,\mu(dx) \text{ and } \mu \circ g(A) = \int_A \frac{d\mu \circ g}{d\mu}(x)\,\mu(dx)$$

so:

$$\mu(A) = \int_A (\frac{d\mu \circ f}{d\mu}(x) + \frac{d\mu \circ g}{d\mu}(x))\,\mu(dx)$$

then $d\mu \circ f/d\mu + d\mu \circ g/d\mu = 1$ for all x, $\mu - a.e.$.

Since $Z(x, f(x)) = Z(x, g(x)) = 0$ for all x it results from the CVR property that $Q(x,.) = \frac{d\mu \circ f}{d\mu}(x)\delta_{f(x)} + \frac{d\mu \circ g}{d\mu}(x)\delta_{g(x)}$ is such that $\tilde{Z}(x, Q(x,.)) = 0$. For Q being a SE we need to prove that $\frac{d\mu \circ f}{d\mu}(x) > 0$ and $\frac{d\mu \circ g}{d\mu}(x) > 0$ for all x, $\mu - a.e.$. For proving this it is sufficient that $\mu << \mu \circ f$ and $\mu << \mu \circ g$, because from the first step these measures will be equivalent. Let $A \in \mathcal{B}(X)$ such that $\mu \circ f(A) = 0$ then $\mu(f(A)) = 0$ and therefore $\lambda(f(A)) = 0$ because of μ is equivalent to λ restricted to the support of μ (which is ϕ-invariant) and we can consider $A \subset supp(\mu)$. By differentiability:

$$\lambda(f(A)) = \int_A |det(f'(z))|\,\lambda(dz)\ ,$$

hence $\lambda(A) = 0$. Then $\lambda(g(A)) = 0$, so by $\mu(A) = \mu \circ f(A) + \mu \circ g(A) = 0$, therefore $\mu << \mu \circ f$ and $\mu << \mu \circ g$.

Finally let us prove the stationarity. For $A \in \mathcal{B}([0,a])$:

$$\int_{[0,a]} Q(x,A)\,\mu(dx) = \int_{[0,a]} \frac{d\mu \circ f}{d\mu}(x)\delta_{f(x)}\mu(dx)\ +\ \int_{[0,a]} \frac{d\mu \circ g}{d\mu}(x)\delta_{g(x)}\,\mu(dx)$$

$$= \int_{[0,a]} 1_{f^{-1}(A)}(x)\,\mu \circ f(dx)\ +\ \int_{[0,a]} 1_{g^{-1}(A)}(x)\,\mu \circ g(dx)$$

$$= \mu \circ f(f^{-1}(A)) + \mu \circ g(g^{-1}(A)) = \mu(A \cap X_1) + \mu(A \cap X_2) = \mu(A).\ \text{Q.E.D.}$$

Proof of lemma 4.1. Note that in this case $\phi : [0, +\infty) \to [0, l_1^*)$ because $v_1 : [0, l_1^*) \to (0, +\infty)$. It is easy to see that $\phi(0) = 0$ and:

$$v_1'(\phi(x))\phi'(x) = d^{-1}v_2'(sd^{-1}x) \tag{2}$$

therefore $\phi'(0) = (d\bar{\theta})^{-1} > 1$. From (2) we can observe that every critical point of v_2 is a critical point of ϕ then (putting $y = sd^{-1}x$) we need to find $y^* > 0$ such that:

$$v_2'(y^*) = V_2'(l_2^* + y^*) + y^* V_2''(l_2^* + y^*) = 0$$

or what is the same:

$$\frac{l_2^* + y^*}{y^*} = R_2(l_2^* + y^*) = \alpha_2 \tag{3}$$

but the leftside of (3) is a strictly decreasing function of y^* which tends to 1 when $y^* \to +\infty$. Then there exists a unique y^* which satisfies (3). Furthermore, from (2):

$$v_2'(y) = V_2''(l_2^* + y)[1 - \frac{y}{l_2^* + y}\alpha_2].$$

Now $V_2'' < 0$ and the term in brackets is strictly decreasing and vanishes at y^*. Therefore y^* is a local (in fact global) maximum. Finally from (2):

$$v_1''(\phi(x))(\phi'(x))^2 + v_1'(\phi(x))\phi''(x) = sd^{-2}v_2''(sd^{-1}x).$$

Replacing $x = x^* = s^{-1}dy^*$ demostrates $\phi''(x^*) < 0$; therefore ϕ is a unimodal map and its critical point is non-flat.

Since $v_1 \circ \phi = s^{-1}v_2 \circ (sd^{-1})$ we obtain from properties of Schwarzian derivative:

$$(Sv_1 \circ \phi)(\phi')^2 + S\phi = (Sv_2 \circ (sd^{-1}))(sd^{-1})^2,$$

hence $S\phi < 0$ and finally $\phi \in \mathcal{F}$.Q.E.D.

Proof of theorem 5.1. From the hypotheses, the invariant measure μ is a B-R-S measure, so for any $I \in [0, a]$:

$$\tau_{x_0,N}(I) \to \mu(I), \text{ when } N \to +\infty; \; x_0 \; \lambda - a.e.$$

But from stationarity $\mu(I) = Prob[\tilde{x}_t \in I)$.

Proof of theorem 6.1. Consider the structural model given in section 4: $v_1(x_t) = E_{\mu_{t+1}}[v_2(\tilde{x}_{t+1})]$ (where we are dropping the constants for simplicity). Let us analise the actual law of motion induced by (2) and (3):

In period $t = 1$ it results $v_1(x_1) = v_2(x_0)$ or $x_1 = \phi(x_0)$. In period $t = 2$ we have:

$$v_1(x_2) = \pi(x_0, x_1)v_2(x_1) + (1 - \pi(x_0, x_1))v_2(x_1 + \epsilon_2) = v_2(x_1) + (1 - \pi(x_0, x_1))v_2'(x_1 + \theta\epsilon_2)\epsilon_2$$

this implies:

$$x_2 = v_1^{-1}(v_2(x_1)) + (1 - \pi(x_0, x_1))v_2'(x_1 + \theta\epsilon_2)(v_1^{-1})'(p)\epsilon_2$$

where p depends on x_0, x_1 and ϵ_2. In general we will have the following dynamics:

$$x_{t+1} = \phi(x_t) + \eta_{t+1}$$

this is the small random perturbation of the dynamical system $x_{t+1} = \phi(x_t)$. Under the assumptions of this theorem, Baladi and Viana (1995) proved that the invariant measure generated by the Markovian process $\tilde{x}_{t+1} = \phi(x_t) + \tilde{\eta}_{t+1}$ converges to the invariant measure μ when the support of the perturbation goes to zero; so the actual conditional probability distributions generated by the actual law of motion converges to $d\mu \circ f/d\mu$. Q.E.D.

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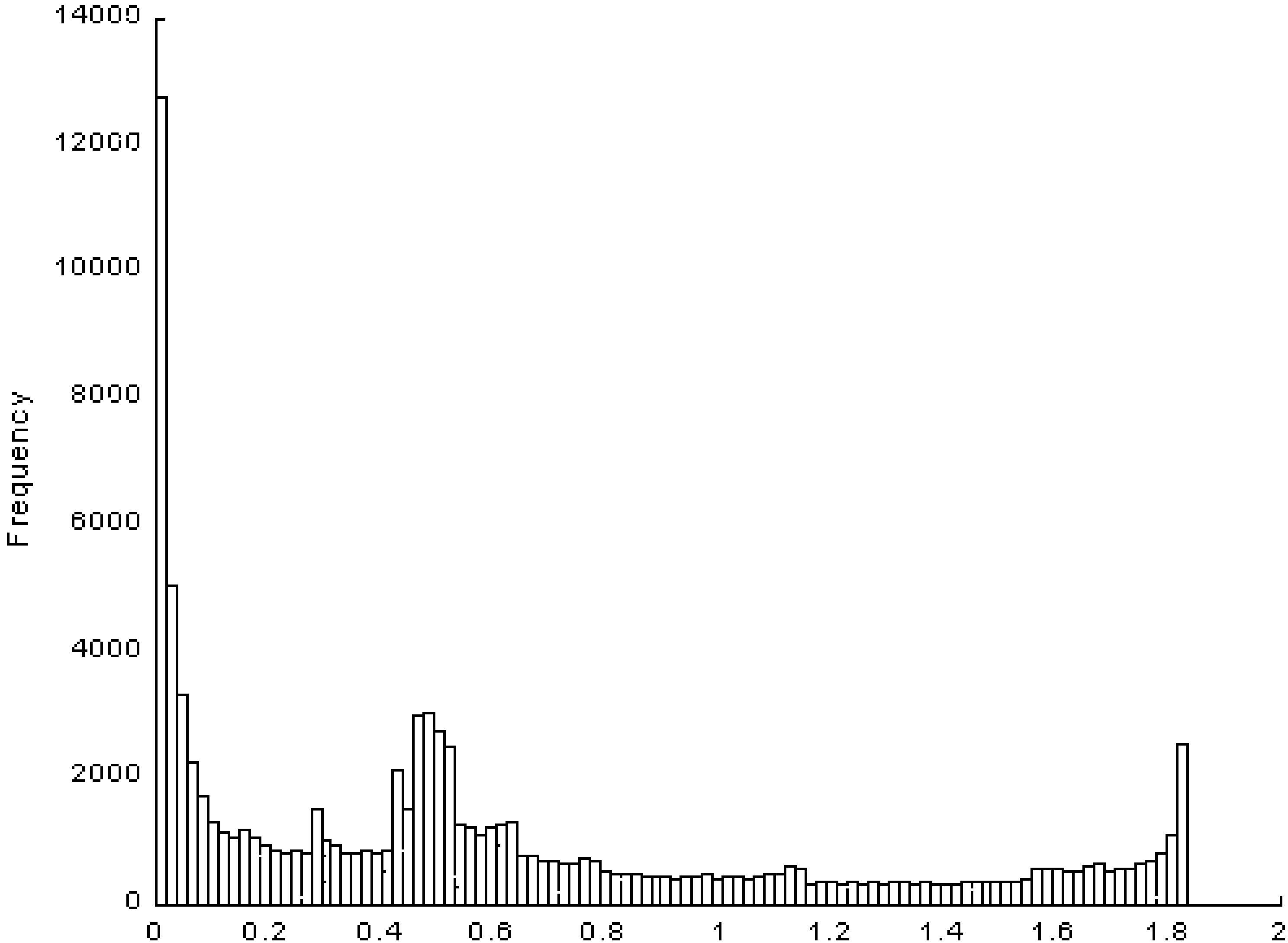
Figure 1:

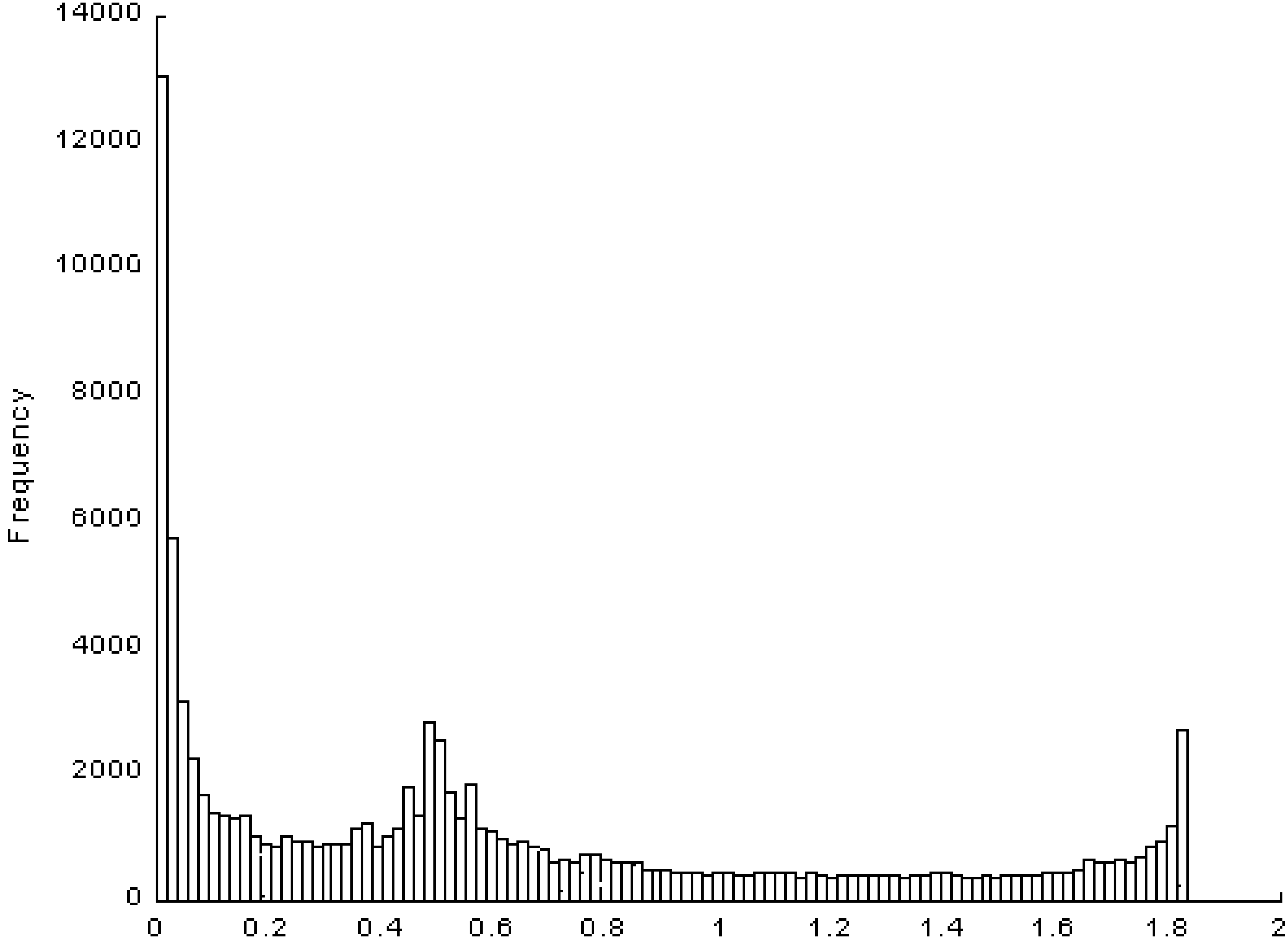


Frequencies of the actual dynamics for the learning process

$$x^e_{t+1} = x_{t-1}$$

Liapunov expoent = 0.3685.

Figure 2:

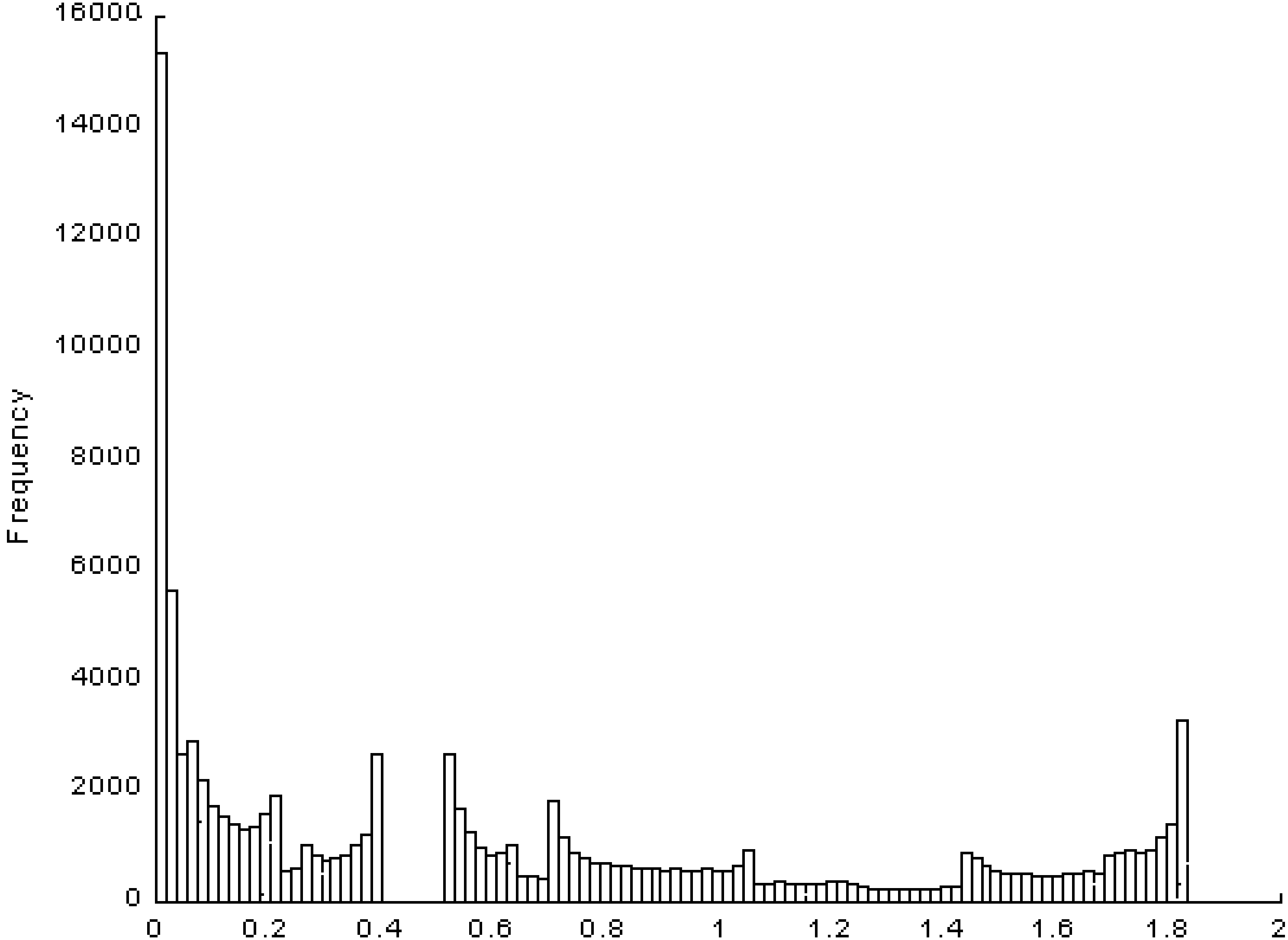


Frequencies of the actual dynamics for the learning process

$$x^e_{t+1} = 0.997x_{t-1} + (1 - 0.997)x^e_{t-1}$$

Liapounov exponent $= 0.3497$ $\quad L^1$ distance to $\mu = 0.1403$

Figure 3:

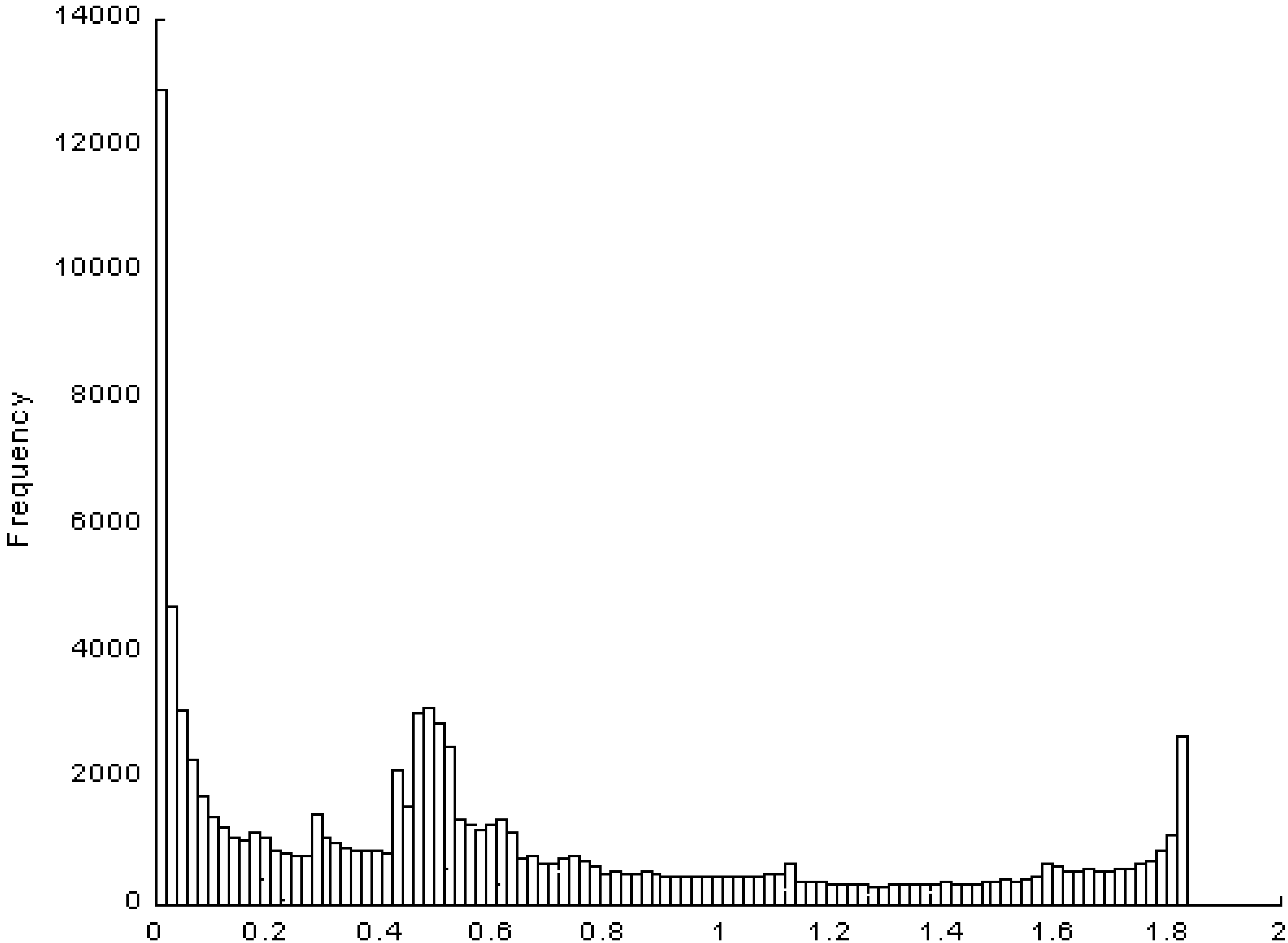


Frequencies of the actual dynamics for the learning process

$$x_{t+1}^e = 0.99x_{t-1} + (1 - 0.99)x_{t-1}^e$$

Liapounov exponent $= 0.2906$ $\quad L^1$ distance to $\mu = 0.4054$

Figure 4:

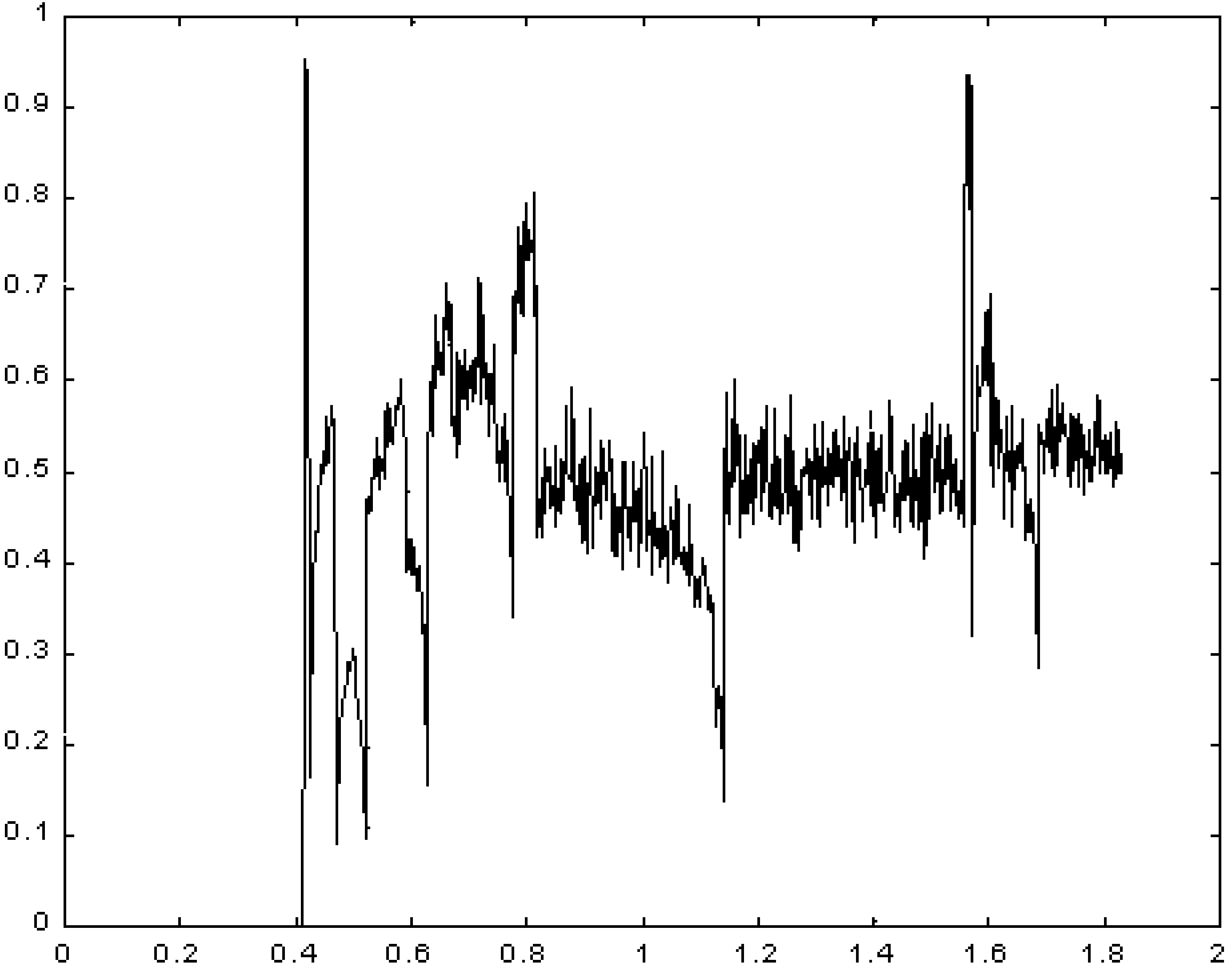


Frequencies of the actual dynamics for the learning process

$$x^e_{t+1} = (1 - (1/t))x_{t-1} + (1/t)x^e_{t-1}$$

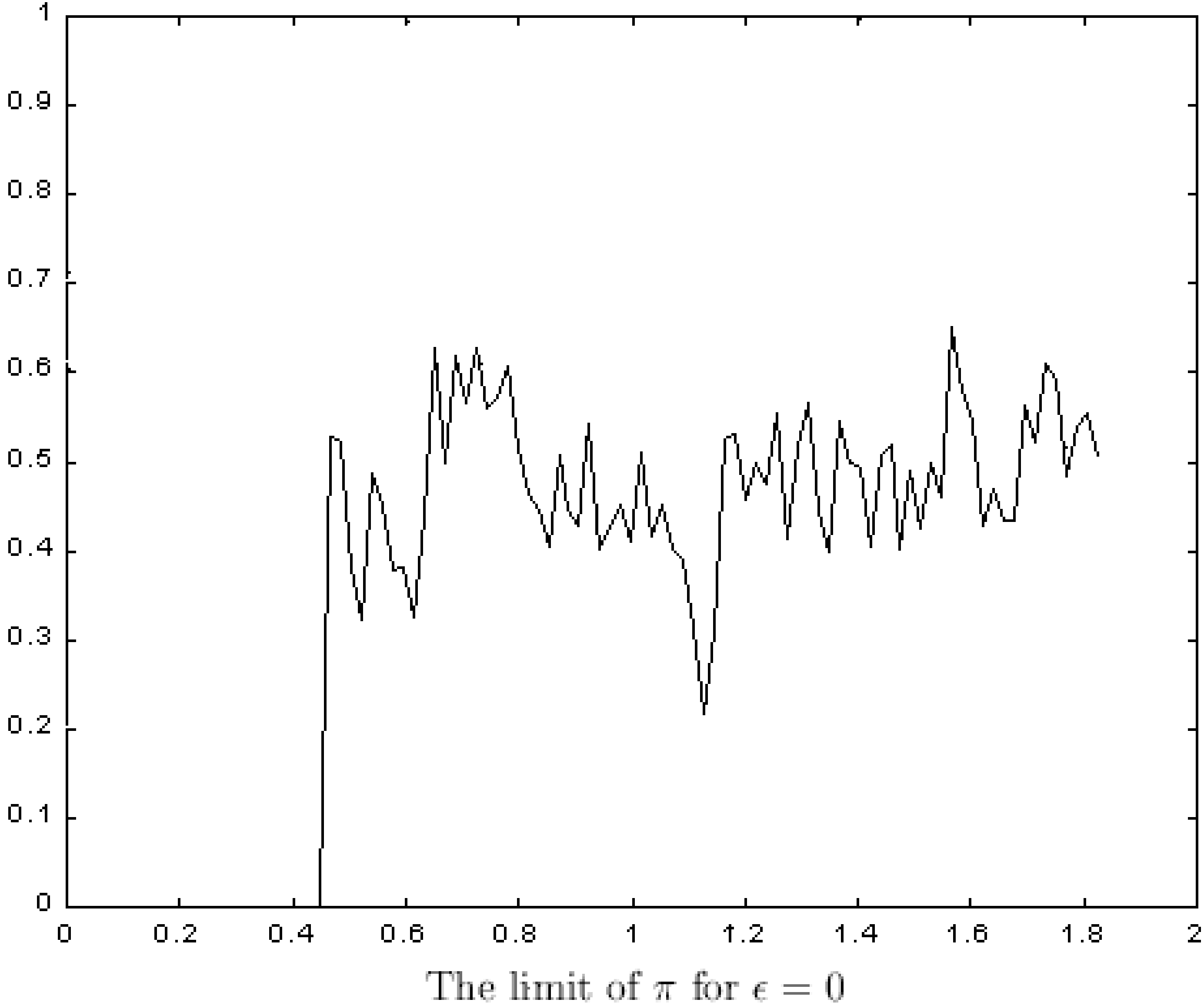
Liapounov exponent $= 0.3666$ $\quad L^1$ distance to $\mu = 0.0497$

Figure 5:



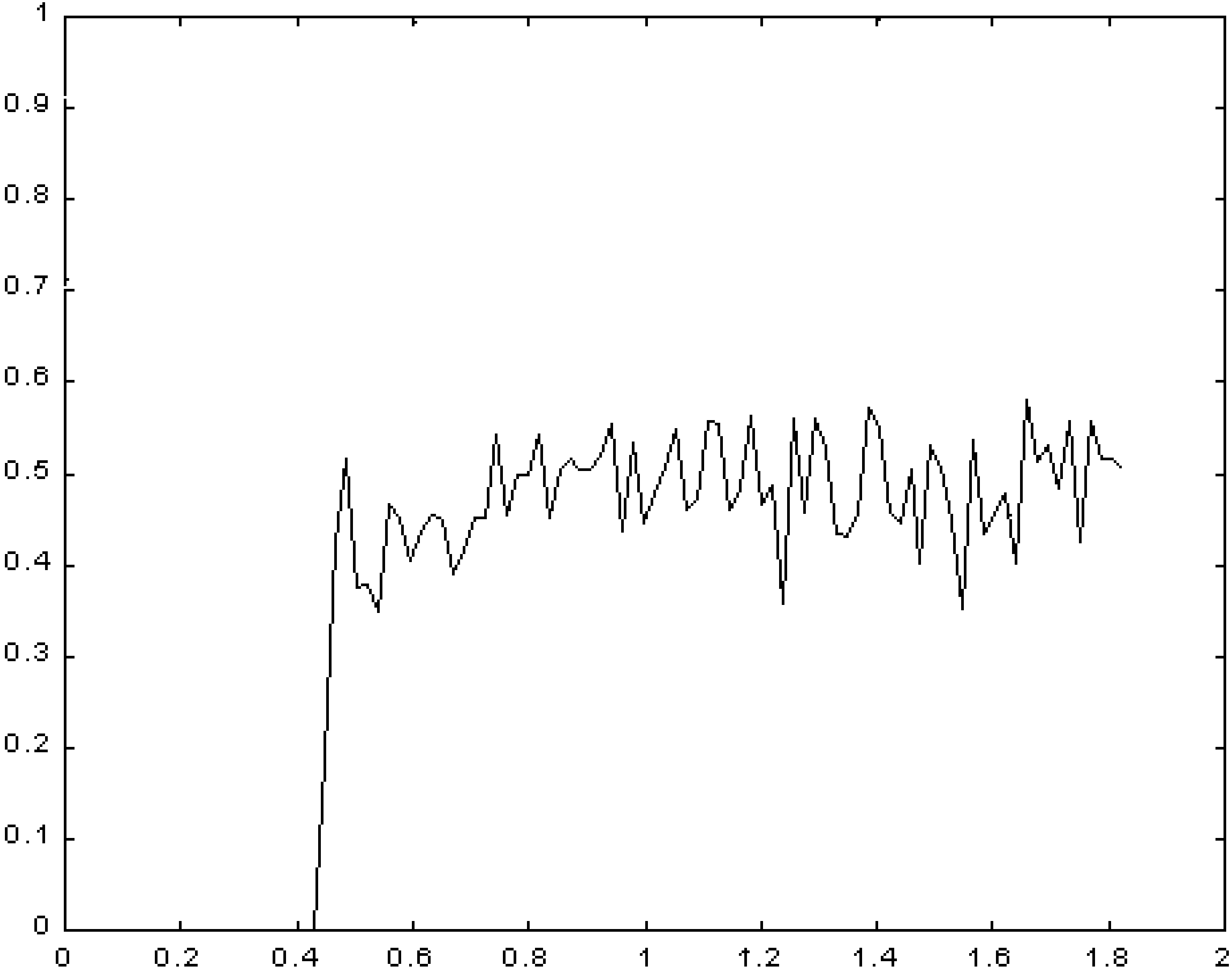
The function $\frac{\partial\mu\circ f}{\partial\mu}$

Figure 6:



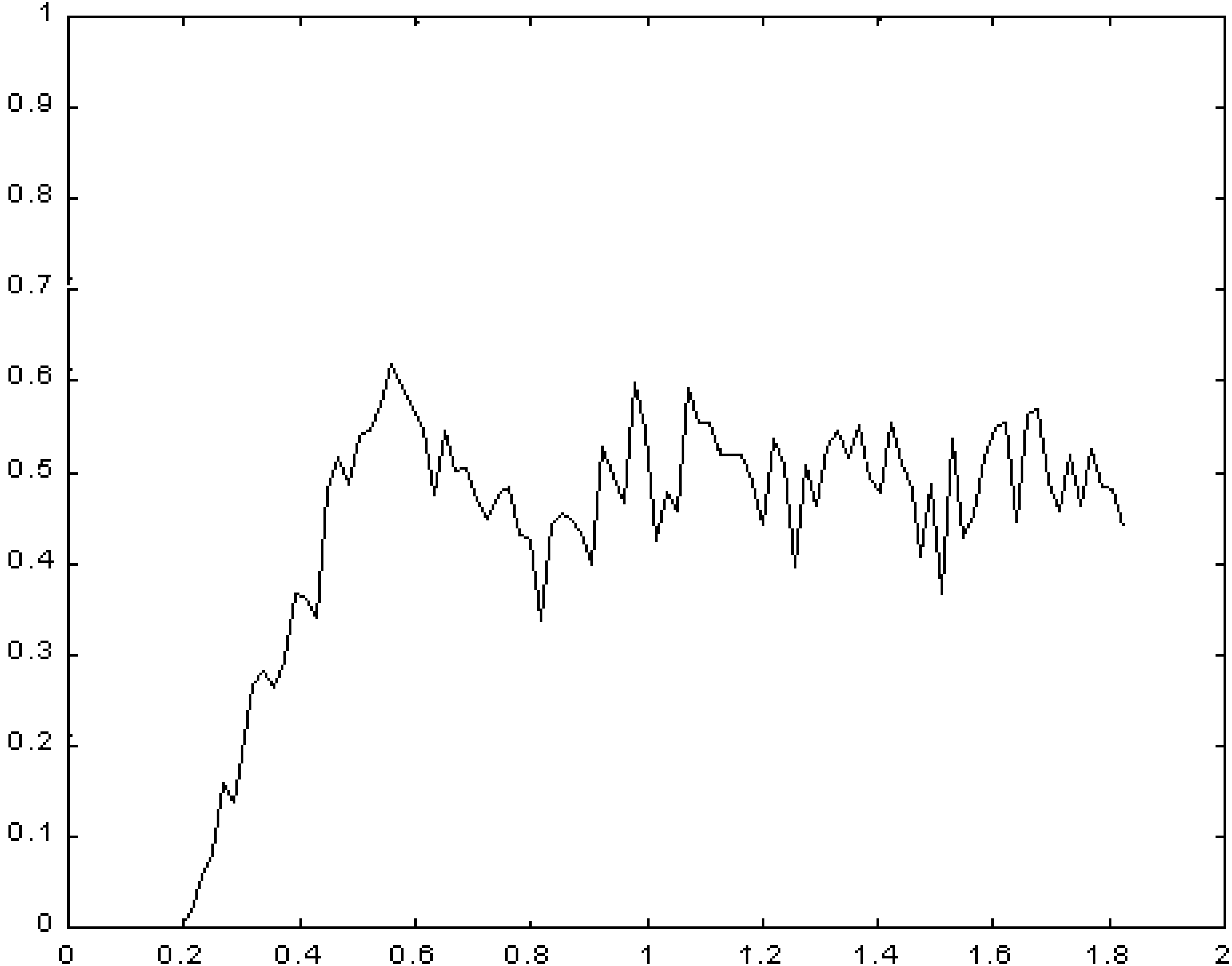
1
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.2
0.4
0.6
0.8
1
1.2
1.4
1.6
1.8
2
The limit of π for $\epsilon = 0$

Figure 7:



The limit of π for $\epsilon = 0.001$

Figure 8:



The limit of π for $\epsilon = 0.01$